Indian Statistical Institute, Bangalore
B. Math. Second Year
First Semester - Analysis III

Mid-Semester Exam | Date : Sept 09, 2014

Answer any five, each question carries 8 marks, total marks: 40

1. (a) Let (f_n) be a sequence of Riemann-integrable functions on $[a,b]$ that converges uniformly to a function f on $[a,b]$. Prove that $f \in \mathcal{R}[a,b]$ and $\int f_n \to \int f$.

   (b) Let (f_n) be a sequence of continuous functions on a compact metric space K. Suppose (f_n) is pointwise bounded and equicontinuous. Prove that (f_n) has a subsequence that converges uniformly on K *(Marks: 4)*.

2. Let (f_n) be a uniformly converging sequence of continuous functions on $\mathbb{R}$.

   (a) If $x_n \to x$, prove that $\lim_{n\to\infty}[f_n(x_n) - f_n(x)] = 0$ *(Marks: 5)*.

   (b) If $f_n(t) = f(nt)$ for some function f on $\mathbb{R}$, prove that f is constant.

3. (a) Let (P_n) be a sequence of real polynomials such that (P_n) converges uniformly to a function f on $\mathbb{R}$. Prove that f is also a polynomial *(Marks: 5)*.

   (b) Let (f_n) be a uniformly convergent sequence of continuous functions on a compact metric space K. Prove that (f_n) is equicontinuous.

4. (a) Let (f_n) be a uniformly bounded sequence of Riemann-integrable functions on $[a,b]$. Define $F_n(x) = \int_a^x f_n(t)dt$, for $x \in [a,b]$. Prove that (F_n) has a subsequence that converges uniformly on $[a,b]$ *(Marks: 4)*.

   (b) Let (f_n) and (g_n) be uniformly converging sequence of functions on a set E. Does $(f_n g_n)$ converge uniformly? Justify your answer.

5. (a) If $\gamma\colon [a,b] \to \mathbb{R}^d$ is smooth, prove that γ is rectifiable and $\Lambda(\gamma) = \int_a^b ||\gamma'(t)||dt$.

   (b) Does $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (3x^2y, x^3y)$ have potential? Justify your answer *(Marks: 3)*.

6. (a) Let ϕ be a differentiable scalar valued function with continuous gradient $\bigtriangledown\phi$ on an open subset S of $\mathbb{R}^d$. If $\alpha\colon [a,b] \to S$ is a piecewise smooth path, prove that $\int \bigtriangledown\phi d\alpha = \phi(\alpha(b)) - \phi(\alpha(a))$ *(Marks: 4)*.

   (b) Let $S = \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{2} < x^2 + y^2 < 2\}$ and $f\colon S \to \mathbb{R}^2$ be $f(x,y) = \frac{1}{x^2+y^2}(-y,x)$. Is the line integral of f independent of paths? Justify your answer.

7. (a) If $\alpha\colon [a,b] \to \mathbb{R}^d$ is a rectifiable (not necessarily smooth) curve, prove that $s\colon [a,b] \to \mathbb{R}$ defined by $s(t) = \Lambda(\alpha|_{[a,t]})$, for $t \in [a,b]$ is continuous *(Marks: 5)*.

   (b) Let f be a continuous function on an open subset S of $\mathbb{R}^d$ such that $\int f d\alpha = c\sum(y_i - x_i)$ for any path α from x to y in S. Prove that $f = c(1,1,\cdots,1)$.